MATH2050C Assignment 12

Deadline: April 18, 2018.

Hand in: 5.4 no. 3, 4, 7; Suppl. Ex. no. 1, 2.

Section 5.4 no. 3, 4, 6-12.

Supplementary Exercise

1. Let f be continuous on (a, b), $-\infty \le a < b \le \infty$. Show that it is uniformly continuous on (a, b) if it is uniformly continuous on $(a, c]$ and $[c, b)$ for some $c \in (a, b)$.

2. Consider $h(x) = 1/x$. Show that it is continuous on $(0, 1]$ by determining the best δ as a function of ε and x_0. And then using it to show h is not uniformly continuous on $(0, 1]$ but uniformly continuous on $[a, 1]$ for any fixed $a \in (0, 1)$. (This was done in class.)

3. Optional. Consider $g(x) = x^{-2}$. Show that it is continuous on $(0, \infty)$ by determining the best δ as a function of ε and x_0. And then using it to show g is not uniformly continuous on $(0, \infty)$ but uniformly continuous on $[a, \infty)$ for any fixed $a > 0$. Suggestion: Show g is uniformly continuous on $[a, 1]$ and $[1, \infty)$ separately.

4. Optional. Let E be a non-empty set in $\mathbb{R}$. Define the distance function (to E) $\rho(x) = \inf\{|z - x| : z \in E\}$. Show that

$$|\rho(x) - \rho(y)| \le |x - y|.$$

The Largest δ for Continuity

Let f be continuous on some nonempty set E in $\mathbb{R}$. When f is continuous at some $x_0 \in E$, it means for each $\varepsilon > 0$, there is some δ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in E, |x - x_0| < \varepsilon$. Here δ in general depending on x_0 and ε. We let

$$\delta(\varepsilon, x_0) = \sup\{\delta : \ |f(x) - f(x_0)| < \varepsilon \ , \quad \forall x \in (x_0 - \delta, x_0 + \delta)\}.$$

It is clear that for $\varepsilon > 0$, $|f(x) - f(x_0)| < \varepsilon$ for all $x \in (x_0 - \delta(\varepsilon, x_0), x_0 + \delta(\varepsilon, x_0))$. Hence $(x_0 - \delta(\varepsilon, x_0), x_0 + \delta(\varepsilon, x_0))$ is the largest (symmetric) interval on which $|f(x) - f(x_0)| < \varepsilon$.

We could define f to be uniformly continuous on E by saying there is a positive lower bound for all $\delta(\varepsilon, x_0)$ for each fixed ε and arbitrary x_0 in E. However, the following reformulation is more convenient.

Let us define f to be **uniformly continuous** on E if for each $\varepsilon > 0$, there is some δ depending on ε only that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in E, |x - y| < \delta$. Note that it follows from definition that any uniformly continuous function is necessarily continuous.

Proposition 1. Let f be continuous on E. It is uniformly continuous if and only if for each ε there is some $\rho > 0$ such that $\delta(\varepsilon, x_0) \geq \rho > 0$.

Proof. When f is uniformly continuous, we simply take ρ to be δ in the definition. On the other hand, if there is such positive lower bound, we simply take δ to be ρ.

Various Ways to establish uniform continuity:

- The main theorem is: Every continuous function on $[a, b]$ must be uniformly continuous.
- Let f be continuous on some interval $(a, b), -\infty \leq a < b \leq \infty$. Then it is uniformly continuous on (a, b) if it is uniformly continuous on $(a, c]$ and $[c, b)$ for some c in between.
- Any linear combination of finitely many uniformly continuous functions is again uniformly continuous.
- The product of two bounded, uniformly continuous functions is again uniformly bounded.
- The composition of two uniformly continuous functions are uniformly continuous.
- A polynomial of degree greater than one is not uniformly continuous on $\mathbb{R}$. On the other hand, a linear function is uniformly continuous on $\mathbb{R}$.

To establish non-uniform continuity we mainly use

Proposition 2 (Nonuniform Continuity Criterion). The function f is not uniformly continuous if and only if there exist some $\varepsilon_0 > 0$ and $\{x_n\}, \{y_n\}$ in E satisfying $|x_n - y_n| \to 0$, but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.